Discrete Mathematics 91 (1991) 283–293
North-Holland

The major index polynomial for conjugacy classes of permutations

Michelle L. Wachs*

Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA

Received 26 August 1988
Revised 19 September 1989

Abstract

Wachs, M.L. The major index polynomial for conjugacy classes of permutations, Discrete Mathematics 91 (1991) 283–293.

Garsia (1988) gives a remarkably simple expression for the major index enumerator for permutations of a fixed cycle type evaluated at a primitive root of unity. He asks for a direct combinatorial proof of this identity. Here we give such a combinatorial derivation.

1. Introduction

We consider a problem posed by Garsia [3] which deals with q-counting permutations with a fixed cycle structure. In [3], a remarkably simple expression for the major index polynomial evaluated at the primitive nth root of unity is derived as a consequence of a result on idempotents for the free Lie algebra. Garsia asks for a direct combinatorial derivation of this identity. In this paper we give such a derivation based on a beautiful combinatorial construction of Gessel [5]. Gessel's construction illuminates the structure of certain subclasses of the class of permutations with a fixed cycle type.

For any partition of n, $\lambda \vdash n$, let C_λ be the conjugacy class of permutations in the symmetric group $\mathcal{S}_n$ whose cycle structure is λ, i.e., permutations whose cycle lengths form the partition λ. Define A_λ to be the polynomial,

$$A_\lambda(q) = \sum_{\sigma \in C_\lambda} q^{\mathrm{maj}(\sigma)},$$

where maj denotes the *major index* statistic. That is, for a permutation $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n$, set $\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{des}(\sigma)} i$, where $\mathrm{des}(\sigma)$ denotes the *descent set* of σ, $\{i \mid \sigma_i > \sigma_{i+1}\}$. Let ω_n be a primitive nth root of unity. We shall give

* Partially supported by a grant from the National Science Foundation.

0012-365X/91/$03.50 © 1991 — Elsevier Science Publishers B.V. (North-Holland)

a direct combinatorial proof of the following identity ([3]):

$$A_\lambda(\omega_n)=\begin{cases}\mu(k) & \text{if } \lambda=k^{n/k},\\ 0 & \text{otherwise,}\end{cases} \tag{1.1}$$

where μ is the classical Möbius function. Note that this says that A_λ is equal, mod the cyclotomic polynomial, to the constant on the right-hand side of (1.1).

In Section 2 we review Gessel's construction. In Section 3 we derive results on the major index polynomial for certain subclasses of C_λ called product classes. In Section 4 we use the formulas of Section 3 to derive (1.1). Section 5 deals with descent set preserving bijections on product classes.

2. Gessel's necklaces

Gessel, in an unpublished work [5], has developed a construction which enables him to derive generating functions for counting permutations by descents, major index and cycle structure (see [4]). This construction is also a useful tool in studying derangements (cf. [4] and [2]). Here, it shall provide the framework for a combinatorial explanation of (1.1). It is closely related to Stanley's theory of P-partitions [7].

We now present Gessel's construction. A *necklace* is defined to be a primitive circular word on the set of nonnegative integers, i.e., an equivalence class under cyclic rotation of a word $a_1a_2\cdots a_k$ which is not a concatenation of several copies of a shorter word. The *length* of a necklace c is the length of the word and the *weight* $|c|$ of a necklace c is the sum of its letters. An *ornament* of order n is a multiset of necklaces whose lengths sum to n. The *type* $\lambda(f)$ of an ornament f is the partition of n formed by the lengths of its necklaces. The *weight* $|f|$ of an ornament f is the sum of the weights of its necklaces.

We say that a (weakly) decreasing sequence $s=(s_1\geqslant s_2\geqslant\cdots\geqslant s_n)$ is *compatible* with $\sigma\in\mathscr{S}_n$ if $s_i>s_{i+1}$ whenever $i\in\mathrm{des}(\sigma)$. Let D_n be the set of decreasing length n sequences of nonnegative integers and let D_σ be the elements of D_n that are compatible with σ. The *weight* $|s|$ of a sequence $s\in D_n$ is the sum of its entries.

Gessel's construction associaties with each ornament f of type λ, a unique pair (σ,s), such that $\sigma\in C_\lambda$, $s\in D_\sigma$, and $|s|=|f|$. The sequence s is obtained by simply arranging the letters of f in weakly decreasing order. By writing a permutation in cycle notation, we can think of a permutation of cycle type λ as an ornament of type λ with distinct letters. To obtain σ from f, we replace the letters of f with n distinct letters from 1 to n as follows. Each position x of the ornament f is associated with the infinite word w_x obtained by starting at position x and repeatedly reading the circular word in a clockwise direction. Note that if $w_x=w_y$ then either $x=y$ or x and y are positions in identical but distinct necklaces of the ornament, since the necklaces are primitive. Fix an ordering for each collection of identical necklaces. The lexicographical ordering of the infinite words together with the fixed ordering of identical necklaces induces a total ordering of the positions of f. This total order is used to assign distinct letters from 1 to n to the positions of f and thereby obtain σ. That is, position x is assigned a letter smaller than that of position y if either w_x is lexicographically greater than w_y, or if $w_x = w_y$ and the necklace containing position x precedes the necklace containing position y.

We illustrate this construction with the following example. Let f be the ornament $(3,3,0)\ (3,1)\ (3,1)\ (3,0)\ (2)$. Label the 10 positions from left to right. We have

$$\begin{aligned}
&w_1 = 330330\cdots, \quad && w_2 = 303303\cdots, \quad && w_3 = 033033\cdots,\\
&w_4 = 313131\cdots, && w_5 = 131313\cdots, && w_6 = 313131\cdots,\\
&w_7 = 131313\cdots, && w_8 = 303030\cdots, && w_9 = 030303\cdots,\\
&w_{10} = 222222\cdots.
\end{aligned}$$

The total ordering of the positions is $1<4<6<2<8<10<5<7<3<9$. Hence position 1 is assigned 1, position 4 is assigned 2, position 6 is assigned 3, etc. The resulting permutation in cycle form is $(1,4,9)\ (2,7)\ (3,8)\ (5,10)\ (6)$. So the ornament f is associated with the pair

$$(\sigma, s) = (4,\ 7,\ 8,\ 9,\ 10,\ 6,\ 2,\ 3,\ 1,\ 5;\ 3,\ 3,\ 3,\ 3,\ 3,\ 2,\ 1,\ 1,\ 0,\ 0),$$

where σ is now written in word form. Note that $\operatorname{des}(\sigma) = \{5, 6, 8\}$ and that s is compatible with σ.

Proposition 2.1 (Gessel [5]). *The above construction is a bijection between the set of ornaments of type λ and the set of pairs (σ, s) where $\sigma \in C_\lambda$ and $s \in D_\sigma$.*

Note that the inverse of the bijection is simply the map that takes (σ, s) to the ornament obtained by writing σ in cycle form and then replacing the integer k by the kth element of s.

From the theory of P-partitions [7], we have the following proposition.

Proposition 2.2. *Let $\sigma \in \mathcal{S}_n$. Then*

$$\sum_{s\in D_\sigma} q^{|s|} = \frac{q^{\operatorname{maj}(\sigma)}}{\prod_{i=1}^{n}(1-q^i)}.$$

Proof. There is a natural bijection $\varphi: D_\sigma \to D_n$ which is obtained as follows. Let $s = (s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n) \in D_\sigma$. To obtain $\varphi(s)$, for each $j \in \operatorname{des}(\sigma)$, subtract 1 from each s_i, $i = 1, 2, \ldots, j$. Clearly φ is a bijection and $|s| = |\varphi(s)| + \operatorname{maj}(\sigma)$.

We now have

$$\sum_{s\in D_\sigma} q^{|s|} = q^{\operatorname{maj}(\sigma)} \sum_{s\in D_n} q^{|s|}.$$

Summing the right-hand side gives the result. □

For $U \subset \mathcal{S}_n$, let $F(U)$ be the set of all ornaments which map, under the bijection of Proposition 2.1, to (σ, s) for some $\sigma \in U$ and $s \in D_\sigma$. We now consider the generating function

$$G_U(q) = \sum_{f \in F(U)} q^{|f|}.$$

By Propositions 2.1 and 2.2, we have

$$G_U(q) = \sum_{\sigma \in U} \sum_{s \in D_\sigma} q^{|s|} = \frac{\sum_{\sigma \in U} q^{\mathrm{maj}(\sigma)}}{\prod_{i=1}^{n} (1 - q^i)}. \tag{2.1}$$

3. Product classes

Let $\mathcal{S}_A$ denote the set of permutations of the set A. For $\alpha \in \mathcal{S}_A$ and $A = \{a_1 < a_2 < \cdots < a_n\}$, the *reduction* of α, is the permutation in $\mathcal{S}_n$ obtained by replacing each letter a_i in α by i, $i = 1, 2, \ldots, n$. Conversely, for $\alpha \in \mathcal{S}_n$ and $|A| = n$, let α^A denote the permutation in $\mathcal{S}_A$ whose reduction is α. For example, the reduction of 5, 8, 7, 3 is 2, 4, 3, 1, and if $\alpha = 2, 4, 3, 1$ *and* $A = \{4, 5, 6, 7\}$ then $\alpha^A = 5, 7, 6, 4$. We shall call two permutations $\alpha \in \mathcal{S}_A$ and $\beta \in \mathcal{S}_B$ *equivalent* if their reductions are equal.

Let $\alpha \in \mathcal{S}_k$ and $\beta \in \mathcal{S}_{n-k}$. We define the *product class* of α and β, denoted $P(\alpha, \beta)$, to be the set of all permutations in $\mathcal{S}_n$ which can be expressed as products of α^A and β^B for some pair of complementary subsets A and B of $\{1, 2, \ldots, n\}$ of cardinality k and $n - k$, respectively. Note that we are viewing α^A and β^B as permutations in $\mathcal{S}_n$ in order to take the product. In this situation, $\alpha^A \cdot \beta^B(i) = \alpha^A(i)$ if $i \in A$, and $\alpha^A \cdot \beta^B(i) = \beta^B(i)$ if $i \in B$. The notion of product classes can clearly be extended to product classes of any multiset of permutations whose lengths sum to n.

It turns out that the major index polynomial for product classes has a particularly nice formulation in terms of Gaussian coefficients. Recall the *Gaussian coefficients* (*q-binomial coefficients* and *q-multinomial coefficients*) are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!} \quad \text{for } 0 \leq k \leq n$$

and

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_r \end{bmatrix} = \frac{[n]!}{[k_1]!\,[k_2]! \cdots [k_r]!} \quad \text{for } 0 \leq k_i \text{ and } k_1 + \cdots + k_r = n,$$

where $[n]$ denotes the polynomial $1 + q + q^2 + \cdots + q^{n-1} = (q^n - 1)/(q - 1)$ and $[n]!$ denotes the polynomial $[n][n-1] \cdots [1]$.

Theorem 3.1. *Suppose $\alpha \in \mathcal{S}_k$, $\beta \in \mathcal{S}_{n-k}$ and α and β have no equivalent cycles. Then*

$$\sum_{\sigma \in P(\alpha,\beta)} q^{\mathrm{maj}(\sigma)} = q^{\mathrm{maj}(\alpha)+\mathrm{maj}(\beta)} \begin{bmatrix} n \\ k \end{bmatrix}.$$

Proof. It is easy to see that each ornament f in $F(P(\alpha, \beta))$, viewed as a multiset of necklaces, is the union of disjoint ornaments f_1 in $F(\{\alpha\})$ and f_2 in $F(\{\beta\})$. Consequently, there is a bijection between $F(P(\alpha, \beta))$ and $F(\{\alpha\}) \times F(\{\beta\})$, such that if f maps to (f_1, f_2) then $|f| = |f_1| + |f_2|$. It follows that

$$G_{P(\alpha,\beta)}(q) = G_{\{\alpha\}}(q) \cdot G_{\{\beta\}}(q).$$

By (2.1), we have

$$\frac{\sum_{\sigma \in P(\alpha,\beta)} q^{\mathrm{maj}(\sigma)}}{\prod_{i=1}^{n} (1-q^i)} = \frac{q^{\mathrm{maj}(\alpha)}}{\prod_{i=1}^{k} (1-q^i)} \frac{q^{\mathrm{maj}(\beta)}}{\prod_{i=1}^{n-k} (1-q^i)},$$

from which the result follows. □

Let $P(\alpha^m)$ denote the product class of m copies of the permutation α, that is, $P(\alpha^m)$ denotes $P(\alpha, \alpha, \ldots, \alpha)$ where α occurs m times.

Theorem 3.2. *For any cycle $\alpha \in C_k$ and $m \geq 0$,*

$$\sum_{\sigma \in P(\alpha^m)} q^{\mathrm{maj}(\sigma)} = q^{m \cdot \mathrm{maj}(\alpha)} B_{m,k}(q),$$

where $B_{m,k}(q)$ is the polynomial defined recursively by,

$$B_{m,k}(q) = \frac{1}{m} \sum_{j=1}^{m} B_{m-j,k}(q) \begin{bmatrix} mk \\ jk \end{bmatrix} \prod_{\substack{i=1 \\ j \nmid i}}^{jk-1} (1-q^i).$$

$$B_{0,k}(q) = 1.$$

Proof. Every ornament in $F(P(\alpha^m))$ is a multiset of m necklaces in $F(\{\alpha\})$. Hence by Proposition 2.1, ornaments in $F(P(\alpha^m))$ correspond bijectively to multisets of m sequences in D_α and the weight of an ornament equals the sum of the weights of the m sequences. We shall view a multiset of sequences as a weakly increasing list of sequences under lexicographical order and use the symbol $\leqslant$ to denote this order. It follows that

$$\begin{aligned} G_{P(\alpha^m)}(q) &= \sum_{\substack{s_1 \leqslant s_2 \leqslant \cdots \leqslant s_m \\ s_i \in D_\alpha}} q^{\sum |s_i|} \\ &= q^{m \cdot \mathrm{maj}(\alpha)} \sum_{\substack{s_1 \leqslant s_2 \leqslant \cdots \leqslant s_m \\ s_i \in D_k}} q^{\sum |s_i|}, \end{aligned} \tag{3.1}$$

with the second equality following from the bijection from D_α to D_k in the proof of Proposition 2.2. The following lemma will allow us to deal with the sum in (3.1).

Lemma. *Let $(X, \leqslant)$ be a totally ordered set and let $f: X^m \to \mathbb{N}$ be a symmetric function in m variables. Then the following formal power series identity holds*:

$$m \sum_{\substack{x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m \\ x_i \in X}} q^{f(x_1, x_2, \ldots, x_m)} = \sum_{j=0}^{m-1} \sum_{\substack{x_1 \leqslant x_2 \leqslant \cdots \leqslant x_j \\ x_i, y \in X}} q^{f(x_1, x_2, \ldots, x_j, y, \ldots, y)}. \tag{3.2}$$

Proof. Since f is symmetric, we can define for any multiset $M = \{x_1, x_2, \ldots, x_m\}$ on X

$$f(M) = f(x_1, x_2, \ldots, x_m).$$

Now (3.2) becomes

$$m \sum_{\substack{M \subset X \\ |M| = m}} q^{f(M)} = \sum_{\substack{M \subset X, y \in X \\ |M| < m}} q^{f(M \cup \{y^{m-|M|}\})}.$$

Each multiset $M \subset X$, $|M| = m$, is represented on the right-hand side as,

$$(M - \{y\}) \cup \{y\}, (M - \{y^2\}) \cup \{y^2\}, \ldots, (M - \{y^{m_y}\}) \cup \{y^{m_y}\},$$

for each distinct y in M, where m_y is the multiplicity of y in M. It follows that $f(M)$ appears m_y times on the right hand side for each distinct y in M. This means that $f(M)$ appears a total of $\sum m_y = m$ times. $\square$

Proof of Theorem 3.2 continued. Applying the lemma to the sum in (3.1) yields

$$\begin{aligned}
\sum_{\substack{s_1 \leqslant s_2 \leqslant \cdots \leqslant s_m \\ s_i \in D_k}} q^{\sum |s_i|} &= \frac{1}{m} \sum_{j=0}^{m-1} \sum_{\substack{s_1 \leqslant s_2 \leqslant \cdots \leqslant s_j \\ s_i, s \in D_k}} q^{\sum |s_i|} q^{(m-j)|s|} \\
&= \frac{1}{m} \sum_{j=0}^{m-1} \sum_{\substack{s_1 \leqslant s_2 \leqslant \cdots \leqslant s_j \\ s_i \in D_k}} q^{\sum |s_i|} \sum_{s \in D_k} q^{(m-j)|s|} \\
&= \frac{1}{m} \sum_{j=1}^{m} \sum_{\substack{s_1 \leqslant s_2 \leqslant \cdots \leqslant s_{m-j} \\ s_i \in D_k}} q^{\sum |s_i|} \frac{1}{\prod_{i=1}^{k} (1 - q^{ji})}.
\end{aligned}$$

On multiplying both sides of the equation by $(1 - q^{mk})(1 - q^{mk-1}) \cdots (1 - q)$ and setting

$$\bar{B}_{m,k}(q) = (1 - q^{mk})(1 - q^{mk-1}) \cdots (1 - q) \sum_{\substack{s_1 \leqslant s_2 \leqslant \cdots \leqslant s_m \\ s_i \in D_k}} q^{\sum |s_i|}, \quad m \geqslant 1$$

$$\bar{B}_{0,k}(q) = 1,$$

we have,

$$\bar{B}_{m,k}(q) = \frac{1}{m}\sum_{j=1}^{m} \bar{B}_{m-j,k}(q)\frac{(1-q^{mk})(1-q^{mk-1})\cdots(1-q^{(m-j)k+1})}{(1-q^{jk})(1-q^{j(k-1)})\cdots(1-q^{j})}$$

$$= \frac{1}{m}\sum_{j=1}^{m} \bar{B}_{m-j,k}(q)\begin{bmatrix} mk \\ jk \end{bmatrix}\prod_{\substack{i=1 \\ j \nmid i}}^{jk-1}(1-q^{i}).$$

Hence $\bar{B}_{m,k}(q)$ satisfies the defining recurrence relation for $B_{m,k}(q)$ and is therefore equal to $B_{m,k}(q)$.

We now replace the sum in (3.1) by $B_{m,k}(q)/(1-q^{mk})(1-q^{mk-1})\cdots(1-q)$ to obtain

$$G_{P(\alpha^m)}(q) = \frac{q^{m\cdot\mathrm{maj}(\alpha)}B_{m,k}(q)}{(1-q^{mk})(1-q^{mk-1})\cdots(1-q)}.$$

The result now follows from (2.1). □

The next result follows inductively from Theorems 3.1 and 3.2.

Theorem 3.3. *For any multiset of cycles $M = \{\alpha_1^{m_2}, \alpha_2^{m_2}, \ldots, \alpha_r^{m_r}\}$, where $\alpha_i \in C_{k_i}$, and $\sum m_i k_i = n$, we have*

$$\sum_{\sigma\in P(M)} q^{\mathrm{maj}(\sigma)} = \begin{bmatrix} n \\ m_1k_1, m_2k_2, \ldots, m_rk_r \end{bmatrix}\prod_{i=1}^{r}\sum_{\sigma\in P(\alpha_i^{m_i})} q^{\mathrm{maj}(\sigma)}$$

$$= q^{\sum m_i\,\mathrm{maj}(\alpha_i)}\begin{bmatrix} n \\ m_1k_1, m_2k_2, \ldots, m_rk_r \end{bmatrix}\prod_{i=1}^{r} B_{m_i,k_i}(q).$$

Theorem 3.1 also implies the following formula for the major index q-polynomial for general conjugacy classes.

Theorem 3.4. *For any partition $\lambda = k_1^{m_1}k_2^{m_2}\cdots k_r^{m_r}$ of n,*

$$A_\lambda(q) = \begin{bmatrix} n \\ m_1k_1, m_2k_2, \ldots, m_rk_r \end{bmatrix}\prod_{i=1}^{r} A_{k_i^{m_i}}(q).$$

Proof. Note that C_λ is the disjoint union $\bigcup P(\alpha_1, \alpha_2, \ldots, \alpha_r)$ which ranges over $\{(\alpha_1, \ldots, \alpha_r) \mid \alpha_i \in C_{k_i^{m_i}}\}$. The result now follows by inductively applying Theorem 3.1. □

Now we need only a formula for $A_{k_i^{m_i}}$. The following corollary of Theorem 3.3 gives a formula that is not very clean, but is nevertheless useful for our purposes.

Corollary 3.5. *Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be any subset of C_k and set $A_{S,m}(q) = \sum q^{\mathrm{maj}(\sigma)}$, where the summation index σ ranges over all permutations in $\mathcal{S}_{mk}$ which* *decompose into m cycles that are equivalent to cycles in S. Then*

$$A_{S,m}(q) = \sum_{\substack{0 \le m_1, \ldots, m_r < m \\ m_1 + \cdots + m_r = m}} q^{\sum m_i \operatorname{maj}(\alpha_i)} \begin{bmatrix} mk \\ m_1 k, m_2 k, \ldots, m_r k \end{bmatrix} \prod_{i=1}^{r} B_{m_i,k}(q)$$
$$+ A_{S,1}(q^m) B_{m,k}(q).$$

Remark. By considering the generating function for necklaces, Gessel [4] obtains a nice formula for $A_k(q)$, which involves the classical Möbius function. This formula will not, however, be needed in our derivation of (1.1).

4. The polynomials at primitive nth roots of unity

Note that when q is set equal to a primitive nth root of unity, ω_n, the multinomial coefficients evaluate to

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_r \end{bmatrix}_{q=\omega_n} = \begin{cases} 0 & \text{if } n_i < n \text{ for all } i, \\ 1 & \text{if } n_1 = n \text{ and } n_2, n_3, \ldots, n_r = 0. \end{cases} \tag{4.1}$$

Lemma 4.1. *For $m \geq 0$ and $k \geq 1$, $B_{m,k}(\omega_{mk}) = 1$.*

Proof. We use (4.1) to eliminate all but one of the terms of the recurrence relation for $B_{m,k}(q)$ given in Theorem 3.2. We are then left with

$$B_{m,k}(\omega_{mk}) = \frac{1}{m} \prod_{\substack{i=1 \\ m \nmid i}}^{mk-1} (1 - \omega_{mk}^i) = \frac{1}{m} \frac{\prod_{i=1}^{mk-1} (1 - \omega_{mk}^i)}{\prod_{i=1}^{k-1} (1 - \omega_{mk}^{mi})}$$
$$= \frac{1}{m} \frac{\prod_{i=1}^{mk-1} (1 - \omega_{mk}^i)}{\prod_{i=1}^{k-1} (1 - \omega_k^i)} = \frac{1}{m} \frac{mk}{k} = 1,$$

since

$$\prod_{i=1}^{n-1} (1 - \omega_n^i) = \prod_{i=1}^{n-1} (x - \omega_n^i) \Big|_{x \to 1} = \frac{x^n - 1}{x - 1} \Big|_{x \to 1} = n. \qquad \square$$

Theorem 4.2. *Let $S \subset C_k$. Then for all $m \geq 0$,*

$$A_{S,m}(\omega_{mk}) = A_{S,1}(\omega_k).$$

Proof. The result is obtained by applying (4.1) and Lemma 4.1 to the formula in Corollary 3.5. $\square$

We are now able to easily prove (1.1). First by (4.1) and Theorem 3.4, we see that $A_\lambda(\omega_n) = 0$ if λ is not of the form $k^{n/k}$. Next we have

$$[n]! = \sum_{\sigma \in \mathscr{S}_n} q^{\operatorname{maj}(\sigma)} = \sum_{\lambda \vdash n} A_\lambda(q).$$

Setting q equal to ω_n, for $n \geqslant 2$, we arrive at

$$\begin{aligned} 0 &= \sum_{\lambda \vdash n} A_\lambda(\omega_n) = \sum_{k \mid n} A_{k^{n/k}}(\omega_n) \\ &= \sum_{k \mid n} A_k(\omega_k) \qquad \text{(by Theorem 4.2).} \end{aligned}$$

For $n = 1$ we have $A_n(\omega_n) = 1$. Since $A_k(\omega_k)$ satisfies precisely the recurrence relation for the classical Möbius function $\mu(k)$, it follows that $A_k(\omega_k) = \mu(k)$ for all $k \geqslant 1$. By Theorem 4.2 we also have that $A_{k^{n/k}}(\omega_n) = \mu(k)$, which completes the proof of (1.1).

5. Bijections, shuffles and descent sets

Theorem 3.1 (and its proof) resembles a result on shuffles of permutations [6, Theorem 3.1]. Let S and T be any pair of complementary subsets of $\{1, 2, \ldots, n\}$ of cardinality k and $n-k$ respectively. Let $\mathrm{Sh}(\alpha^S, \beta^T)$ be the set of all shuffles of α^S and β^T, i.e., permutations in $\mathcal{S}_n$ which contain α^S and β^T as complementary subwords. The formula for the major index q-polynomial for $\mathrm{Sh}(\alpha^S, \beta^T)$ given in [6] is identical to the one given here for $P(\alpha, \beta)$. Hence the distribution of major index is identical for both classes of permutations. A stronger result can, in fact, be proved: The number of permutations with a given descent set is identical for both classes of permutations.

In [9], for α a derangement and β the identity, a direct descent set preserving bijection φ between $P(\alpha, \beta)$ and $\mathrm{Sh}(\tilde{\alpha}, \tilde{\beta})$ is given, where α and $\tilde{\alpha}$ have the same length, $\mathrm{des}(\alpha) = \mathrm{des}(\tilde{\alpha})$, and the same conditions hold for β and $\tilde{\beta}$. It would be interesting to extend this bijection to more general α and β. Here, we give a direct bijection between $P(\alpha, \beta) \cap D^J$ and $\mathrm{Sh}(\alpha^S, \beta^T) \cap D^J$ for any α, β with no equivalent cycles and any $J \subset \{1, 2, \ldots, n-1\}$ (D^J is the set of all permutations with descent set contained in J). This can be used, via the involution principle, to construct a descent set preserving bijection between $P(\alpha, \beta)$ and $\mathrm{Sh}(\alpha^S, \beta^T)$.

Let $J = \{j_1 < j_2 < \cdots < j_d\} \subset \{1, 2, \ldots, n-1\}$ and let $\tilde{\alpha} = \alpha^S$ and $\tilde{\beta} = \beta^T$. For $\sigma \in P(\alpha, \beta) \cap D^J$, let A, B be such that σ is the product of α^A and β^B. To construct $\pi = \varphi(\sigma) \in \mathrm{Sh}(\tilde{\alpha}, \tilde{\beta})$, first let $\pi_1, \pi_2, \ldots, \pi_{j_1}$ be the unique increasing sequence obtained by merging the increasing sequences $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{i_1}$ and $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_{k_1}$, where $i_1 = |A \cap \{1, 2, \ldots, j_1\}|$ and $k_1 = |B \cap \{1, 2, \ldots, j_1\}|$. Next, let $\pi_{j_1+1}, \ldots, \pi_{j_2}$ be the unique increasing sequence obtained by merging the increasing sequences $\tilde{\alpha}_{i_1+1}, \ldots, \tilde{\alpha}_{i_1+i_2}$ and $\tilde{\beta}_{k_1+1}, \ldots, \tilde{\beta}_{k_1+k_2}$, where $i_2 = |A \cap \{j_1+1, j_1+2, \ldots, j_2\}|$ and $k_2 = |B \cap \{j_1+1, j_1+2, \ldots, j_2\}|$. Continuing in this way gives π. Although the map φ is quite easy to describe, it is not at all obvious that it is a bijection. Gessel's construction enables us to prove that it is.

Theorem 5.1. *The map* $\varphi : P(\alpha, \beta) \cap D^J \to \mathrm{Sh}(\tilde{\alpha}, \tilde{\beta}) \cap D^J$ *described above is a bijection.*

Proof. Clearly φ is a well defined map. The proof that it is a bijection relies on Proposition 2.1, via the following lemma.

Lemma 5.2. *Let M be the multiset $\{d^{j_1}, (d-1)^{j_2-j_1}, \ldots, 0^{n-j_d}\}$, which we shall also view as a weakly decreasing sequence in D_n. Then the map*

$$\psi : P(\alpha, \beta) \cap D^J \to \{(M_1, M_2) \mid M_1 \in D_\alpha, M_2 \in D_\beta, M_1 \cup M_2 = M\},$$

defined by

$$\psi(\sigma) = (M_A, M_B),$$

where $\sigma = \alpha^A \cdot \beta^B$ and M_A and M_B are the subsequences of M determined by positions in A and B, respectively, is a bijection.

Proof. It is easy to that the map ψ is well defined. We show that ψ is a bijection by describing it in terms of Gessel's bijection (Proposition 2.1). Let $\sigma \in P(\alpha, \beta) \cap D^J$. Since $M \subset D_\sigma$, by Proposition 2.1, there is a unique ornament f in $F(P(\alpha, \beta))$ corresponding to the pair (σ, M). This ornament decomposes uniquely into the union of ornaments $f_1 \in F(\{\alpha\})$ and $f_2 \in F(\{\beta\})$. Again under Gessel's bijection f_1 corresponds to (α, M_1) and f_2 corresponds to (β, M_2). Note that $M_1 = M_A$ and $M_2 = M_B$. Hence this process of going from σ to (M_1, M_2) via Gessel's bijection gives ψ. Since the process is clearly reversible, ψ is invertible and is therefore a bijection. □

Proof of Theorem 5.1 continued. Let M be as in Lemma 5.2. Consider the map

$$\gamma : \{(M_1, M_2) \mid M_1 \in D_\alpha, M_2 \in D_\beta, M_1 \cup M_2 = M\} \to \mathrm{Sh}(\tilde{\alpha}, \tilde{\beta}) \cap D^J$$

defined by $\gamma(M_1, M_2) = \pi$ where π is constructed as follows: Let $\pi_1, \pi_2, \ldots, \pi_{j_1}$ be the unique increasing sequence obtained by merging the increasing sequences $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{i_1}$ and $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_{k_1}$, where i_1 and k_1 are the multiplicities of d in M_1 and M_2, respectively. Then let $\pi_{j_1+1}, \ldots, \pi_{j_2}$ be the unique increasing sequence obtained by merging the increasing sequences $\tilde{\alpha}_{i_1+1}, \ldots, \tilde{\alpha}_{i_1+i_2}$ and $\tilde{\beta}_{k_1+1}, \ldots, \tilde{\beta}_{k_1+k_2}$, where i_2 and k_2 are the multiplicities of $d-1$ in M_1 and M_2, respectively. Continuing this way gives π. It is easy to check that γ is a bijection. One can also see that $\varphi = \gamma \circ \psi$. Hence by Lemma 5.2, φ is a bijection □

An actual descent set preserving bijection between $P(\alpha, \beta)$ and $\mathrm{Sh}(\tilde{\alpha}, \tilde{\beta})$ can now be recursively constructed via a variation of the Garsia–Milne involution principle (see [8, Sec 2.6]). The map ψ can also be used to construct, via the involution principle, a descent set preserving bijection between $P(\alpha, \beta)$ and $P(\tilde{\alpha}, \tilde{\beta})$, where $\mathrm{des}(\alpha) = \mathrm{des}(\tilde{\alpha})$, $\mathrm{des}(\beta) = \mathrm{des}(\tilde{\beta})$, α and β have no equivalent cycles, and $\bar{\alpha}$ and $\bar{\beta}$ have no equivalent cycles. Also similar arguments enable us to construct a descent set preserving bijection between $P(\alpha^m)$ and $P(\bar{\alpha}^m)$, where α, $\bar{\alpha} \in C_k$ and $\mathrm{des}(\alpha) = \mathrm{des}(\bar{\alpha})$. A direct (not using the involution principle) descent set preserving bijection for shuffles is constructed in [1]. It would be interesting to find such a bijection for product classes, as well.

Acknowledgement

This work was initiated while I was visiting the University of California, San Diego during the summer of 1987. I would like to thank Adriano Garsia for bringing his problem to my attention and informing me of Gessel's construction. A nice alternative proof of Theorem 3.2 involving the generating function for the complete homogeneous symmetric functions has been given by the referee.

References

[1] A. Björner and M.L. Wachs, q – Hook length formulas for forests, J. Combin. Theory Ser. A 52 (1989) 165–187.
[2] J. Désarménien and M.L. Wachs, Descents des dérangements mot circulaires, Séminaire Lotharingien de Combinatoire 19 (1988) 13–21.
[3] A.M. Garsia, Combinatorics of the free Lie algebra and the symmetric group, in: P. Rabinowitz and E. Zehnder, eds., Analysis Et Cetera (Academic Press, New York, 1990) 309–382.
[4] I. Gessel, Counting permutations by descents, greater index, and cycle structure, MIT, preprint.
[5] I. Gessel, personal communication.
[6] A.M. Garsia and I. Gessel, Permutation statistics and partitions, Adv. in Math. 31 (1979) 288–305.
[7] R.P. Stanley, Ordered structures and partitions, Memoirs Amer. Math. Soc. 119 (1972).
[8] R.P. Stanley, Enumerative Combinatorics, Vol. 1 (Wadsworth, Belmont, CA, 1986).
[9] M.L. Wachs, On q-derangement numbers, Proc. Amer. Math. Soc. 106 (1989) 273–278.